CONVEXITY AND APPROXIMATION OF NONLINEAR GAUSSIAN CHANCE CONSTRAINTS

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Optimal power flow

$$\underset{p,\theta}{\text{minimize}} \sum_{i \in \mathcal{G}} c_i p_i$$

subject to

$$\sum_{n \in \mathcal{B}} B_{bn}\theta_n = \sum_{i \in G_b} p_i + w_b^f - d_b, \quad \forall b \in \mathcal{B},$$
$$p_i^{min} \le p_i \le p_i^{max}, \quad \forall i \in \mathcal{G},$$
$$f_{mn} = \beta_{mn}(\theta_m - \theta_n), \quad \forall \{m,n\} \in \mathcal{L},$$
$$-f_{mn}^{max} \le f_{mn} \le f_{mn}^{max}, \quad \forall \{m,n\} \in \mathcal{L},$$

where w_b^f is (uncertain) contribution from wind.

- Min-cost network flow governed by physical transmission constraints
- Above model is "DC approximation" to nonlinear AC powerflow laws

Proposal: generators implement a *proportional response policy* to random deviations in the wind forecast $\mathbf{\Omega}$.

$$\boldsymbol{p}_i = p_i - \alpha_i \mathbf{\Omega}$$

- "Supply = Demand" always satisfied if $\sum_i \alpha_i = 1$.

CHANCE CHANCE CONSTRAINT MODEL (BIENSTOCK ET AL., 2014)

Proposal: generators implement a *proportional response policy* to random deviations in the wind forecast $\boldsymbol{\Omega}$.

$$\boldsymbol{p}_i = p_i - \alpha_i \boldsymbol{\Omega}$$

- “Supply = Demand” always satisfied if $\sum_i \alpha_i = 1$.

We then impose that the random line flows $\boldsymbol{f}_{mn}$ stay within limits with high probability:

$$\mathbb{P}(\boldsymbol{f}_{mn} \leq f_{mn}^{max}) \geq 1 - \epsilon$$
$$\mathbb{P}(\boldsymbol{f}_{mn} \geq -f_{mn}^{max}) \geq 1 - \epsilon$$

- Natural to treat these as soft constraints, no immediate consequences if violated

Line flow chance constraints can be expressed as linear chance constraints of the form

$$\mathbb{P}_{\xi}(x^T\xi \leq b) \geq 1 - \epsilon. \tag{1}$$

If ξ is jointly Gaussian with known mean and covariance and $\epsilon \leq \frac{1}{2}$, then (1) is representable as a single **second-order cone** constraint, convex in x and b.

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Gaussian assumption can be relaxed by introducing uncertainty sets on mean and covariance (Bienstock et al., 2014; Lubin et al., 2015)

Voltage-aware optimal power flow (Chertkov)

$$\underset{p,\theta}{\text{minimize}} \sum_{i \in \mathcal{G}} c_i p_i$$

subject to

$$\sum_{n \in \mathcal{B}} B_{bn}\theta_n = \sum_{i \in G_b} p_i + w_b^f - d_b, \quad \forall b \in \mathcal{B},$$

$$\sum_{n \in \mathcal{B}} B_{bn}v_n = \sum_{i \in G_b} q_i + w_b^{fq} - d_b^q, \quad \forall b \in \mathcal{B},$$

$$p_i^{min} \leq p_i \leq p_i^{max}, \quad \forall i \in \mathcal{G},$$

$$q_i^{min} \leq q_i \leq q_i^{max}, \quad \forall i \in \mathcal{G},$$

$$f_{mn}^p = \beta_{mn}(\theta_m - \theta_n), \quad \forall \{m,n\} \in \mathcal{L},$$

$$f_{mn}^q = \beta_{mn}(v_m - v_n), \quad \forall \{m,n\} \in \mathcal{L},$$

$$(f_{mn}^p)^2 + (f_{mn}^q)^2 \leq (f_{mn}^{max})^2, \quad \forall \{m,n\} \in \mathcal{L},$$

Active and reactive power flow on the same physical lines, transmission is limited by the norm,

$$(f^p_{mn})^2 + (f^q_{mn})^2 \leq (f^{max}_{mn})^2, \quad \forall \{m, n\} \in \mathcal{L},$$

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Using a similar model to account for uncertainty in wind, we end up with a chance constraint of the form

$$\mathbb{P}_\xi \left((a^T\xi + b)^2 + (c^T\xi + d)^2 \leq k\right) \geq 1 - \epsilon,$$

where a, b, c, d are decision variables.

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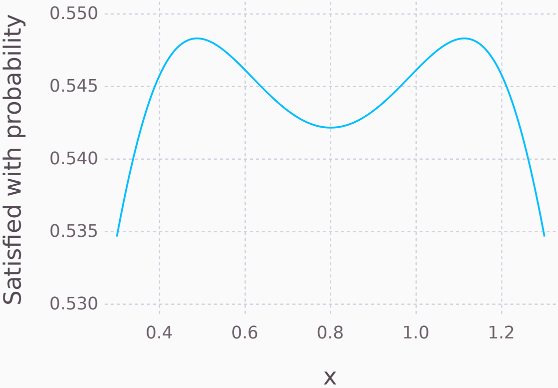
Is this a convex constraint??

Not convex for $\epsilon = 0.445$

$$P((x\xi_1)^2 + (y\xi_2)^2 \leq 1) \geq 1 - \epsilon$$



Not convex for $\epsilon = 0.445$

$$P((x\xi_1)^2 + (y\xi_2)^2 \leq 1) \geq 1 - \epsilon$$

- Counterexample does not apply for smaller ϵ, but anyway let's look for approximations

- **Simpler constraint** $\mathbb{P}_{\xi}\left(|a^T\xi + b| \leq k\right) \geq 1 - \epsilon$ **is convex**
- Theoretically tractable by separation oracles, we give **SOCP approximation with provable approximation guarantee**.
- Using these absolute value constraints, **we obtain a conservative approximation to the quadratic chance constraint**
- Favorable comparison with alternative approximations via **robust optimization** and **CVaR** (Nemirovski and Shapiro, 2007)

Define
$S_\epsilon := \{(x, y) : \int_x^y \varphi(t)\, dt \geq 1 - \epsilon\} = \{(x, y) : \Phi(y) - \Phi(x) \geq 1 - \epsilon\}$.

Then the set S_ϵ is convex for $\epsilon \in (0, 1)$.

Proof: Left-hand side is log-concave.

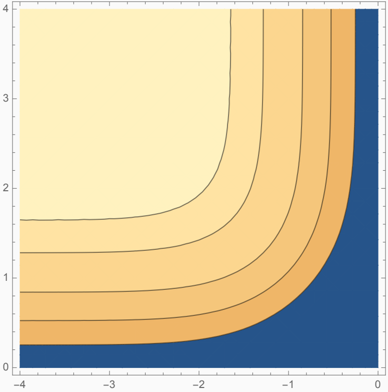


Figure: S_ϵ for $\epsilon = 0.6, 0.7, 0.8, 0.9, 0.95$

Let

$$\begin{aligned}\bar{S}_\epsilon &:= \mathrm{cl}\{(x, y, z) : (x/z, y/z) \in S_\epsilon, z > 0\} \\ &= \mathrm{cl}\{(x, y, z) : \Phi(y/z) - \Phi(x/z) \geq 1 - \epsilon, z > 0\} \\ &= \mathrm{cone}(S_\epsilon \times \{1\})\end{aligned}$$

be the conic hull of S_ϵ. Then $\bar{S}_\epsilon$ is convex.

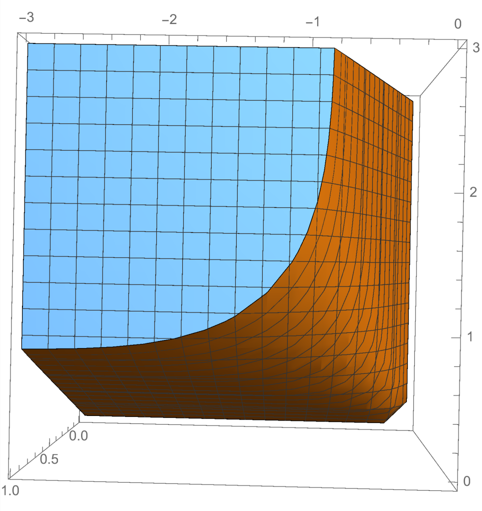


Figure: $\bar{S}_\epsilon$

Let ξ be a vector of i.i.d. standard Gaussian random variables and $0 < \epsilon \leq \frac{1}{2}$. Then the set

$$\{(a, b, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \mathbb{P}(|a^T\xi + b| \leq k) \geq 1 - \epsilon\}$$

is convex.

$$\{(a, b, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \mathbb{P}(|a^T\xi + b| \leq k) \geq 1 - \epsilon\}$$

Proof:

$$\mathbb{P}(|a^T\xi + b| \leq k) \geq 1 - \epsilon$$

iff

$$\mathbb{P}(-k - b \leq a^T\xi \leq k - b) \geq 1 - \epsilon$$

iff

$$\mathbb{P}\left(\frac{-k - b}{||a||} \leq \frac{a^T\xi}{||a||} \leq \frac{k - b}{||a||}\right) \geq 1 - \epsilon$$

iff

$$(-k - b, k - b, ||a||) \in \bar{S}_\epsilon$$

iff $\left(\epsilon \leq \frac{1}{2}\right)$

$$\exists\, t \geq ||a|| \text{ such that } (-k - b, k - b, t) \in \bar{S}_\epsilon$$

$$\{(a, b, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \mathbb{P}(|a^T\xi + b| \leq k) \geq 1 - \epsilon\} =$$
$$\text{proj}_{a,b,k} \left\{(a, b, k, t) : t \geq ||a||, (-k - b, k - b, t) \in \bar{S}_\epsilon \right\}$$

- Representation using standard SOC constraint plus nonstandard “Gaussian integral” cone.
- Theoretically tractable via separation oracles
- Don’t know of any solvers which support as is

$$\{(a,b,k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \mathbb{P}(|a^T\xi + b| \le k) \ge 1-\epsilon\} =$$
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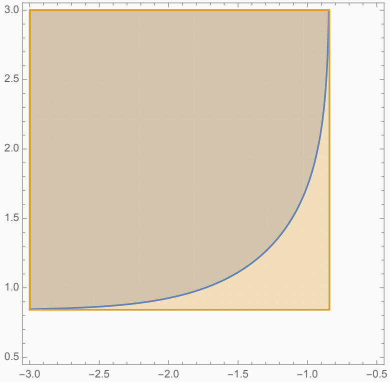
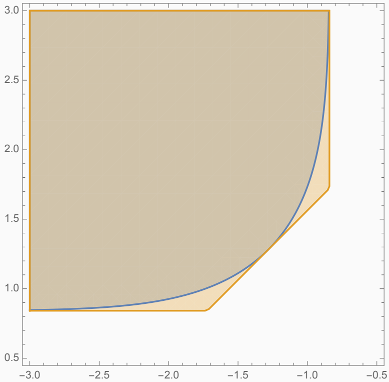
- Representation using standard SOC constraint plus nonstandard “Gaussian integral” cone.
- Theoretically tractable via separation oracles
- Don’t know of any solvers which support as is
- **Will develop polyhedral approximation of** $\bar{S}_\epsilon$

2-CONSTRAINT OUTER APPROXIMATION OF S_ϵ



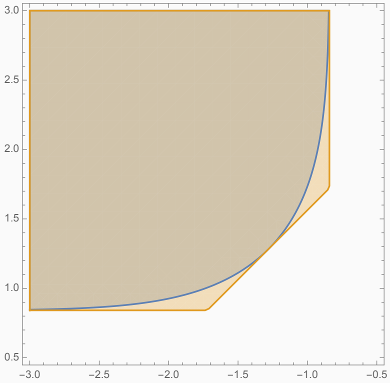
- With two linear constraints, we guarantee that chance constraint holds with 2ϵ. (Can be made conservative.)
- **Proof**: split into two linear chance constraints $\mathbb{P}(a^T\xi + b \leq k) \geq 1 - \epsilon$, $\mathbb{P}(a^T\xi + b \geq -k) \geq 1 - \epsilon$

3-CONSTRAINT OUTER APPROXIMATION OF S_ϵ



- With **three** linear constraints, we guarantee that chance constraint holds with 1.25ϵ.
- **Proof**:

3-CONSTRAINT OUTER APPROXIMATION OF S_ϵ



- With **three** linear constraints, we guarantee that chance constraint holds with 1.25ϵ.
- **Proof**: see the paper

Fix $\epsilon < \frac{1}{2}$ and $\beta \in (0,1)$. If $\exists\, f_1, f_2$ such that

$$\mathbb{P}(|a^T\xi + b| \leq f_1) \geq 1 - \beta\epsilon$$
$$\mathbb{P}(|c^T\xi + d| \leq f_2) \geq 1 - (1-\beta)\epsilon$$
$$f_1^2 + f_2^2 \leq k$$

then

$$\mathbb{P}\left((a^T\xi + b)^2 + (c^T\xi + d)^2 \leq k\right) \geq 1 - \epsilon.$$

Proof:

Fix $\epsilon < \frac{1}{2}$ and $\beta \in (0,1)$. If $\exists\, f_1, f_2$ such that

$$\begin{aligned} &\mathbb{P}(|a^T\xi + b| \le f_1) \ge 1 - \beta\epsilon \\ &\mathbb{P}(|c^T\xi + d| \le f_2) \ge 1 - (1-\beta)\epsilon \\ &f_1^2 + f_2^2 \le k \end{aligned}$$

then

$$\mathbb{P}\left((a^T\xi + b)^2 + (c^T\xi + d)^2 \le k\right) \ge 1 - \epsilon.$$

Proof: Union bound

Proposed a convex, tractable (via SOCP) approximation for

$$\mathbb{P}\left((a^T\xi + b)^2 + (c^T\xi + d)^2 \leq k\right) \geq 1 - \epsilon.$$

What about other approaches?

Proposed a convex, tractable (via SOCP) approximation for

$$\mathbb{P}\left((a^T\xi + b)^2 + (c^T\xi + d)^2 \leq k\right) \geq 1 - \epsilon.$$

What about other approaches?

- Robust optimization
- CVaR

Choose suitable uncertainty set $\mathcal{U}$ and enforce

$$(a^T\zeta + b)^2 + (c^T\zeta + d)^2 \leq k, \forall \zeta \in \mathcal{U}.$$

If $\mathcal{U}$ is ellipsoidal, tractable via SDP (Ben Tal et al., 2009).

- Little guidance on choosing $\mathcal{U}$ to enforce chance constraint under Gaussian distribution.
- Naive approach: choose such that $P(\xi \in \mathcal{U}) = 1 - \epsilon$.

“CVaR” approximation

Proposed by Nemirovski and Shapiro (2007), rewrite constraint as

$$\mathbb{E}\left[I((a^T\xi + b)^2 + (c^T\xi + d)^2 - k)\right] \leq \epsilon,$$

where $I(t) = 1$ if $t \geq 0$ and 0 otherwise. Any convex upper bound on I yields a convex conservative approximation. “CVaR” approximation constructs the **best convex upper bound** on I.

- Convex, but membership requires multidimensional integration
- Misconception: **not the “best” convex approximation to the original constraint**

1. "Two-sided" approximation via absolute value chance constraints (SOCP)
2. Robust optimization approximation (SDP)
3. CVaR (Multidimensional integration)

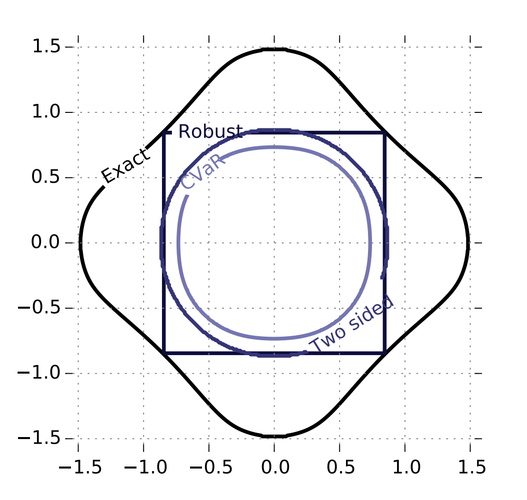


Figure: $P((x\xi_1)^2 + (y\xi_2)^2 \leq 1) \geq 1 - \epsilon$, $\epsilon = 0.5$

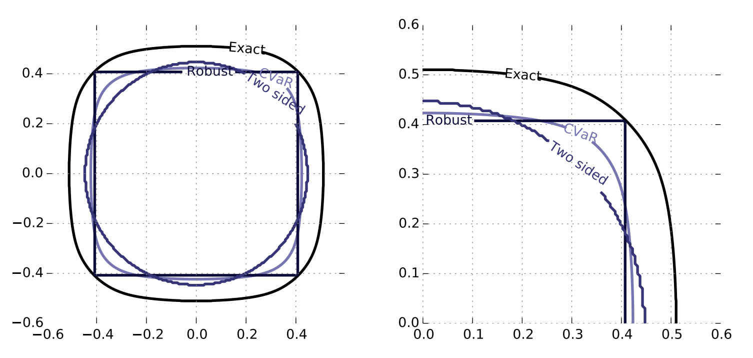


Figure: $P((x\xi_1)^2 + (y\xi_2)^2 \leq 1) \geq 1 - \epsilon$, $\epsilon = 0.05$

- No approximation dominates another, but “two-sided” is most tractable
- `JuMPChance` modeling extension for JuMP supports these constraints
- Power systems case study underway with Y. Dvorkin and L. Roald

`http://arxiv.org/abs/1507.01995`

Thanks! Questions?